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# Exploring Vacuum Manifold of Open String Field Theory

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**Abstract**

The global symmetry generated by $K_n$ is a subgroup of the stringy gauge symmetry. We explore the part of the vacuum manifold related by this symmetry. A strong evidence is presented that the analytic classical solutions to the cubic string field theory found earlier in Refs. [5, 6] are actually related by the symmetry and, therefore, all of them describe the same tachyon vacuum. Some remaining subtlety is pointed out.

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# 1 Introduction

String field theory (SFT) is expected to provide a framework where distinct string backgrounds can be studied in terms of some universal set of underlying degrees of freedom. For example, it was conjectured that SFT admits the closed string vacuum solution describing the state after all unstable D-branes are allowed to decay via condensation of the tachyon [1, 2]. If it happens, no open string excitations appear around the tachyon vacuum, and the energy difference between the originally defined D-brane vacuum and the tachyon vacuum is precisely equal to the brane tension [3, 4].

It is highly desirable to find analytic candidates for the classical solution describing the tachyon vacuum. They will make it possible not only to justify the above conjecture but to give valuable insights into the structure of the vacuum state in SFT. Such a candidate was discussed in Refs. [5, 6]. There, given was a series of one parameter families of classical solutions associated with the functions $h^l_a(w)$. The solutions were constructed with these functions which specify combinations of the BRS current and the ghost field operated on the identity state. They are labeled by an integer $l$, and classified in such a way that non-trivial solutions emerge for the function $h^l_{a_b}(w)$ with the boundary value $a = a_b$, while all other solutions associated with $h^l_a(w)$ $(a \neq a_b)$ become gauge transformations of the D-brane vacuum. In addition to these, we have constructed another class of solutions with higher order zeros [7].

Based on the analysis on the cohomology [6, 7], scattering amplitudes [8] and the potential height [7, 9] around each solution, we understand that all of the above non-trivial solutions are equally valid candidates for the tachyon vacuum. This makes us to believe that all of them are actually equivalent and related by appropriate symmetry transformations present in the SFT action. To clarify this point is the main motivation of this paper.

The SFT action has the gauge symmetry [10] and it is also invariant under transformations generated by particular combinations of the Virasoro operators, $K_n = L_n - (-)^n L_{-n}$ [11, 12, 13]. Later we will see that the symmetry generated by $K_n$ belongs to the "global" part of the gauge symmetry. The solutions, if they are really equivalent, are to be related by some gauge transformations. In this paper, we find operators that transform the higher $l$ solutions down to the $l = 1$ solution. This is the main result of this paper. The operators are written in terms of generators $K_n$ with even $n$. One may wonder what would happen to the inverse of the relations. When we go back to a higher $l$ solution from a lower $l$ solution, we encounter a subtlety, that will be explained in a later section and further discussion will be found in the

last section.

This paper is organized as follows. In the next section, we briefly review how analytic classical solutions to the SFT action are constructed in Refs. [5, 6]. The action expanded around a solution carries a new BRS charge in its kinetic term. The charge seems to have a peculiar ghost structure. However, it will be explained that this is naturally understood from the first quantization point of view. In section 3, we discuss the symmetries of the SFT action. In particular, the deformation of a classical solution under $K_n$-transformations are described. Since the interaction vertex is invariant under the transformation, the change of classical solution has an effect only on the kinetic term, or the BRS charge, in the expanded action. Therefore, when we have actions expanded around two classical solutions that are related by this type of symmetry, we ought to be able to find an appropriate transformation between two BRS charges. This is the subject of the section 4. We construct the operator $U$ relating the BRS charges for the $l > 1$ and $l = 1$ classical solutions and some properties of the operator are reported. The operator has a well-defined normal ordered expression and generate a sound string field transformation on the component fields. The last section is devoted to discussions. We have added four appendices. The appendix A is to evaluate the action for the string field configuration in a pure gauge form. Some technical points in relation to section 3 are explained in appendix B. In appendix C, matrix elements of $U$ are calculated. The universal solutions due to [5, 6] are not in the Siegel gauge as shown in appendix D.

# 2 Classical Solutions in String Field Theory

This section is a short summary of how analytic classical solutions are constructed and what makes us believe that they really correspond to the tachyon vacuum. Though the construction of solutions for SFT looks quite non-trivial, the first quantization point of view provides us with an intuitive picture of the solutions: it also helps us to understand the constraint embedded in the new BRS charge, defined on a non-trivial solution. The first quantization point of view will be explained in the second subsection.

## 2.1 Classical solutions and gauge transformations

We summarize our construction of the classical solutions, paying special attention to gauge transformations in cubic string field theory (CSFT).

The action in the original CSFT [10],[1]

$$S[\Psi, Q_B] = -\frac{1}{g^2} \int \left( \frac{1}{2} \Psi * Q_{\rm B} \Psi + \frac{1}{3} \Psi * \Psi * \Psi \right), \tag{2.1}$$

is characterized by the BRS charge $Q_{\rm B}$ and the $*$-product, which enjoy the following properties:

1. $\int Q_B A = 0$;
2. $Q_B(A * B) = (Q_B A) * B + (-)^{|A|} A * Q_B B$;
3. $(A * B) * C = A * (B * C)$;
4. $\int A * B = (-)^{|A||B|} \int B * A$.

By $|A|$, we denote the Grassmannian even-oddness of the string field $A$: $|A| = +1$ $(-1)$, when A is Grassmann even (odd).

The action (2.1) is an analogue to the integration of the Chern-Simons three form [10, 11]: the string field $\Psi$, the integration $\int$ and the BRS charge $Q_{\rm B}$ correspond to the connection $A$, the integration of differential forms $\int$ and the exterior derivative $d$ of ordinary differential geometry. Similarly to the Chern-Simons action, it is easy to show that the action (2.1) transforms as

$$S[\Psi', Q_B] = S[\Psi, Q_B] + S[g^{-1} * Q_{\rm B} g, Q_B] \tag{2.2}$$

under the gauge transformation

$$\Psi' = g^{-1} * Q_{\rm B} g + g^{-1} * \Psi * g. \tag{2.3}$$

Here, the string functional $g$ is an element of the stringy gauge group in which the multiplication law is given by the star product. This stringy gauge group is expected to have much richer structure than that of the ordinary Yang-Mills theory. Eq. (2.2) reminds us of a similar expression for the Chern-Simon theory (based on a compact group). Though it is tempting to think of a concept of homotopy class for this stringy gauge group, we do not have much more to discuss along this direction. For the purpose of the present paper, it is suffice to know that the second term in eq. (2.2) vanishes for the functional $g$ connected to the identity $I$ via a continuous deformation (see appendix A for the proof). In other words, the action is invariant under such a gauge transformation.

[1] For later convenience, we write the BRS charge dependence of the action as $S[\Psi, Q_B]$.

The equation of motion is given by the variational principle,

$$Q_{\rm B}\Psi + \Psi * \Psi = 0. \tag{2.4}$$

The lhs of (2.4) is in the form of the "field strength". Therefore, at least formally, "flat connections" are classical solutions to string field theory. Let us write such a classical solution as

$$\Psi_0 = g_0^{-1} * Q_{\rm B} g_0, \tag{2.5}$$

where $g_0$ is a (group-valued) string functional.

If we expand the string field around the classical solution (2.5) as

$$\Psi = g_0^{-1} * Q_{\rm B} g_0 + \Phi, \tag{2.6}$$

the action (2.1) becomes

$$S[\Psi, Q_B] = S[g_0^{-1} * Q_{\rm B} g_0, Q_B] - \frac{1}{g^2}\int \left(\frac{1}{2}\Phi * Q'_{\rm B}\Phi + \frac{1}{3}\Phi * \Phi * \Phi\right), \tag{2.7}$$

where the new BRS charge $Q'_{\rm B}$ is defined as

$$Q'_{\rm B}A \equiv Q_{\rm B}A + g_0^{-1} * Q_{\rm B} g_0 * A - (-1)^{|A|} A * g_0^{-1} * Q_{\rm B} g_0, \tag{2.8}$$

for an arbitrary string field $A$.

After having considered the general structure of the string field theory action, let us explicitly describe the classical solutions given in Refs.[5, 6].

Consider the specific gauge functional [5, 6, 8]

$$\begin{aligned} g_0(h) &= \exp(-q_{\rm L}(h) I) \\ &= I - q_{\rm L}(h) I + \frac{1}{2!}\, q_{\rm L}(h) I * q_{\rm L}(h) I + \cdots, \end{aligned} \tag{2.9}$$

where the operator $q_{\rm L}$ is defined in terms of the ghost number current $J_{\rm gh}(w)$ and a function $h(w)$ satisfying $h(\pm i) = 0$ and $h(-1/w) = h(w)$:

$$q_{\rm L}(h) = \int_{C_{\rm left}} \frac{dw}{2\pi i} h(w)\, J_{\rm gh}(w). \tag{2.10}$$

The integration path indicated by the subscript $C_{\rm left}$ is over the left half of the string, ie, $-\pi/2 < \sigma < \pi/2$ on the unit circle for the variable, $w = e^{i\sigma}$. The gauge functional (2.9) gives rise to a classical solution $\Psi_0(h) = g_0(h)^{-1} * Q_{\rm B}\, g_0(h)$, which may be rewritten as

$$\Psi_0(h) = Q_{\rm L}(e^h - 1) I - C_{\rm L}((\partial\, h)^2\, e^h) I, \tag{2.11}$$

where the operators $Q_{\rm L}$ and $C_{\rm L}$ are defined with the BRS current and the ghost field:

$$Q_{\rm L}(f) = \int_{C_{\rm left}} \frac{dw}{2\pi i} f(w)\, J_{\rm B}(w), \quad C_{\rm L}(f) = \int_{C_{\rm left}} \frac{dw}{2\pi i} f(w)\, c(w). \tag{2.12}$$

The solution (2.11) has a well-defined Fock space expression in the universal subspace, spanned by the matter Virasoro generators and ghost oscillators acting on the $SL(2,R)$ invariant vacuum. Therefore, it would be appropriate to call the solution (2.11) as the universal solution.

For the solution (2.11), the new BRS charge can be expressed as

$$Q'_{\rm B} = Q(e^h) - C((\partial h)^2 e^h). \tag{2.13}$$

The operators $Q$ and $C$ are defined as

$$Q(f) = \oint \frac{dw}{2\pi i} f(w)\, J_{\rm B}(w), \quad C(f) = \oint \frac{dw}{2\pi i} f(w)\, c(w). \tag{2.14}$$

Here the integrations are over the unit circle.

Formally, the solution (2.11) is in a pure gauge form, and, therefore, could be gauged away. As shown in [5], however, this is not always the case: some non-trivial solutions emerge at the boundary of one parameter deformation of a certain class of functions chosen for $h$. The functions are given by [5, 6]

$$h_a^l(w) = \log\left\{1 - \frac{a}{2}(-1)^l \left(w^l - \left(-\frac{1}{w}\right)^l\right)^2\right\} \quad (l = 1, 2, 3, \cdots). \tag{2.15}$$

We have a series of functions labeled by the integer $l$. Accordingly, we have a series of non-trivial solutions associated with the functions. The solutions will be addressed as TTK solutions in this paper.

Now we describe evidences suggesting that the TTK solutions are really non-trivial. The reality of the function, for $w$ on the unit circle, restricts the parameter $a$ to be $a \geq -1/2$. This condition guarantees the hermiticity of the corresponding classical solution as well as the new BRS charge, as expected from eq. (2.13). Since $h_{a=0}^l(w) = 0$ and the corresponding classical solution vanishes, it would be reasonable to expect that the solution written with $h_a^l$ is a trivial pure gauge for sufficiently small $a$. Indeed, we know the following facts for $a > -1/2$:

1. The expanded action can be transformed back to the action (2.1) [5];

2. The new BRS charge provides us with the same cohomology as the original BRS charge [5, 6, 7];

3. The expanded theory reproduces ordinary open string scattering amplitudes [8];

4. A numerical study with the level truncation technique shows that the expanded theory has a non-perturbative vacuum and its vacuum energy tends to the D-brane tension as the truncation level increases [9].

These facts are consistent with our expectation that the solution is trivial pure gauge. However, at the boundary $a = -1/2$, the expanded theory shows completely different properties[2]:

5. The new BRS charge has the vanishing cohomology in the Hilbert space with the ghost number one [6, 7];

6. The vanishing of open string scattering amplitudes, the result is consistent with the absence of open string excitations (*no open string theorem*) [8];

7. We can show numerically that the non-perturbative vacuum found for $a > -1/2$ disappears as the parameter $a$ approaches to $-1/2$ [9].

The above results implies that the solution with $a = -1/2$ indeed corresponds to the tachyon vacuum.

## 2.2 Interpretation of new BRS charges in the first quantized theory

The emergence of the non-trivial theory for $a = -1/2$ can also be seen from the first quantization point of view.

In the original action (2.1), we have the Kato-Ogawa's BRS charge

$$Q_{\rm B} = \oint \frac{dw}{2\pi i} \left[c(w)T_X(w) + (bc\,\partial c)(w)\right], \tag{2.16}$$

where $T_X(w)$ is the stress tensor for the string coordinates $X$ and $b$ is the antighost. The BRS charge $Q_{\rm B}$ is known to be constructed directly from the first-class constraint $T_X(w) \approx 0$ [14].

We now consider a modification of the constraint surface by multiplying a function, $e^{h(w)}$. Then, the modified BRS charge constructed from the constraint $e^{h(w)}\, T_X(w) \approx 0$ takes of the form

$$Q'_{\rm B} = \oint \frac{dw}{2\pi i}\, e^{h(w)} \left[c(w)T_X(w) + (bc\,\partial c)(w) + \frac{1}{2}c(w)\{(\partial\, h)^2 + 3(\partial^2\, h)\}\right]. \tag{2.17}$$

[2]It is not known whether the non-trivial solutions with $a = -1/2$ can still be written in the form $g^{-1} * Q_{\rm B} g$. If it is the case, we may consider that the solutions are obtained by "large gauge transformations." That would be a strong evidence for some topological structure of the stringy gauge group.

Here the term linear in the ghost is needed to ensure the nilpotency condition $(Q'_{\rm B})^2 = 0$. The expression (2.17) coincides with (2.13) where the BRS current is given by $j_B(w) = c(w)[T_X(w) + (b\partial c)(w) + 3/2\partial^2 c(w)]$. It means that the replacement of the BRS charge $Q_{\rm B}$ by $Q'_{\rm B}$ corresponds to the replacement of the constraint $T_X(w) \approx 0$ by $e^{h(w)}\, T_X(w) \approx 0$. This change of the constraint can be absorbed by a redefinition of ghost and antighost:

$$\begin{aligned} c(w) \to c(w)e^{h(w)} &= e^{q(h)}c(w)e^{-q(h)}, \\ b(w) \to b(w)e^{-h(w)} &= e^{q(h)}b(w)e^{-q(h)}, \end{aligned} \tag{2.18}$$

so that

$$e^{q(h)}Q_{\rm B}e^{-q(h)} = Q'_{\rm B}. \tag{2.19}$$

This relation holds, of course, only if the operator $e^{q(h)}$ with

$$q(h) = \oint \frac{dw}{2\pi i}\, h(w)\, J_{\rm gh}(w) \tag{2.20}$$

is well-defined. The operator $e^{q(h^l_a)}$, with the function given in (2.15), is well-defined for $a > -1/2$, but not for $a = -1/2$ [5, 6]. Whether the similarity transformation (2.19) makes sense or not depends on the distribution of zeros [15, 16] of the function $\exp(h^l_a(w))$: all zeros are distributed off the the unit circle $|w| = 1$ for $a > -1/2$, while they merge on the unit circle for $a = -1/2$. This change in the distribution of zeros may be related to the non-trivial modification of the constraint in first quantized theory: the constraint surface given by $\exp(h^l_{-1/2}(w))\; T_X(w) \approx 0$ becomes physically distinct from the original surface $T_X(w) \approx \exp(h^l_{a>-1/2}(w))\; T_X(w) \approx 0$.

In this section, the properties of our classical solutions are summarized. As we have seen, there present an infinite number of non-trivial solutions and various results suggest that they all describe the tachyon vacuum. If it is really the case, they are equivalent with each other and related via the symmetries of CSFT. The symmetries of CSFT are the subject of the next section.

# 3 Symmetries of CSFT and Classical Solutions

The CSFT action has a subalgebra of the Virasoro algebra as its symmetry [11, 12, 13]. Here we will see that this symmetry may be considered as a subgroup of the "global" part of the

stringy gauge symmetry. Therefore it provides a way to relate classical solutions and, thereby the SFT actions defined on them.

The subalgebra is generated by $K_n = L_n - (-)^n L_{-n}$, where $L_n$ is the Virasoro operator. Using the properties [11, 12, 13],

$$\begin{aligned} \int K_n A &= 0 \quad \text{for all } A, \\ K_n(A * B) &= (K_n A) * B + A * K_n B, \end{aligned} \tag{3.1}$$

it is easy to show the invariance of the action under an infinitesimal transformation generated by $K_n$ :

$$\delta S \propto \int K_n\Big(\frac{1}{2}\Psi * Q_B\Psi + \frac{1}{3}\Psi * \Psi * \Psi\Big) = 0. \tag{3.2}$$

We now consider a particular type of gauge transformation and find it to be a finite form of $K_n$-transformation written as

$$\Psi' \equiv e^{K(v)}\Psi, \tag{3.3}$$

with $K(v) \equiv \sum_{n>0} v_n K_n$. The parameters $v_n$ will be specified below.

Let us take $u_L(f) \equiv \exp\Big(-\mathcal{T}_L(f) I\Big)$ as a gauge functional. The operator $\mathcal{T}_L(f)$ is defined as

$$\mathcal{T}_{\rm L}(f) = \int_{C_{\rm left}} \frac{dw}{2\pi i}\, f(w)\, T(w), \tag{3.4}$$

similarly to eq. (2.12), with the total energy-momentum tensor $T(w)$. It is easy to see that the gauge transformation (2.3) with $u_L(f)$ can be written as

$$\Psi' = u_L^{-1} * \Psi * u_L = U(f)\Psi, \tag{3.5}$$

$$U(f) \equiv \exp\Big(\oint \frac{dw}{2\pi i}\, f(w)\, T(w)\Big). \tag{3.6}$$

Note that the first term in (2.3) with the BRS charge is absent in the above expression since $[Q_B,\ \mathcal{T}_L(f)] = 0$ and $Q_B I = 0$. In deriving the last expression of eq. (3.5), we used the properties of the half splitting operators in eqs. (B.6) and (B.8) that require the function $f(w)$ to satisfy the condition, $f(w) = (dw/d\tilde{w}) f(\tilde{w})$ for $\tilde{w} = -1/w$.[3] Expanding the function as $f(w) = \sum_n v_n w^{n+1}$, we find the relation, $v_n = v_{-n}(-)^{n+1}$, from the condition. Using this relation, it is easy to see that the integral in eq. (3.6) becomes the operator $K(v) \equiv \sum_{n>0} v_n K_n$.

The absence of the term $u_L(f) * Q_B u_L(f)^{-1}$ in the expression (3.5) allows us an interesting interpretation of the transformation (3.3). The BRS charge $Q_B$ corresponds to the external derivative $d$ in the Chern-Simons theory. A gauge transformation in the CS-theory with

[3]Some more detailed derivation of (3.5) is described in appendix B.

the absence of the derivative term is simply a global transformation. Similarly, a stringy gauge transformation without the first term in (2.3) may be considered to be a "stringy global transformation" in SFT. Note that, strictly speaking, a global transformation is not necessarily a finite $K_n$-transformation. The type of global transformations written as (3.3) forms a specific subgroup in the stringy global transformations. In the rest of the paper, we discuss only this subgroup of the global symmetry.

Now, we apply the global transformations discussed above on a given classical solution. Let us find the action for the fluctuation $\Phi$ around a classical solution $\Psi_0$. Substituting $\Psi \equiv \Psi_0 + \Phi$ into the action $S[\Psi,\ Q_B]$, we obtain

$$S[\Psi_0 + \Phi, Q_B] \quad = \quad S[\Psi_0, Q_B] + S[\Phi, Q_B(\Psi_0)]. \tag{3.7}$$

Here $Q_B(\Psi_0)$ is defined as

$$Q_B(\Psi_0)A \quad \equiv \quad Q_B A + \Psi_0 * A - (-)^{|A|} A * \Psi_0 \tag{3.8}$$

on a string field $A$. The nilpotency follows from the equation of motion for $\Psi_0$.

Once we find a classical solution, we can, at least formally, obtain other solutions related by the gauge symmetry. As for solutions related as eq. (3.3), it is easy to see the following statements to hold:

1. If $\Psi_0$ solves the string equation of motion, ie, $Q_B\Psi_0 + \Psi_0 * \Psi_0 = 0$, then $\Psi_0' \equiv e^{K(v)}\Psi_0$ is also a solution;

2. Furthermore the BRS charges defined around two solutions are related as

$$e^{-K(v)}\ Q_B(e^{K(v)}\Psi_0)\ e^{K(v)} = Q_B(\Psi_0). \tag{3.9}$$

Since the transformation (3.3) leaves the action invariant, the first statement is trivial. Technically speaking, it can be shown by using the property,

$$(e^{K(v)}A) * (e^{K(v)}B) = e^{K(v)}(A * B), \tag{3.10}$$

which is obtained from eq. (3.1). The second statement follows from the relation,

$$Q_B(\Psi_0')e^{K(v)}A = e^{K(v)}Q_B(\Psi_0)A, \tag{3.11}$$

on a generic string field $A$. Eq. (3.11) may be derived from the definition of the BRS charge given in eq. (3.8).

Now we would like to see how a universal solution may be deformed by the action of the $K_n$ operators. Leaving discussions of the finite transformation in the next section, here we consider the change of a solution under an infinitesimal transformation and discuss its implication.

The generic form of the classical solutions obtained in Refs. [5, 6] is given as

$$|\Psi_0\rangle \;=\; Q_L(F)|I\rangle + C_L(G)|I\rangle, \tag{3.12}$$

$$G(w) = -\frac{(\partial F(w))^2}{1+F(w)}, \tag{3.13}$$

where $F(z)$ is an analytic function satisfying the relations $F(-1/w) = F(w)$ and $F(\pm i) = 0$.

It is easy to see that the action of $e^{K(\varepsilon)}$ on the universal solution $|\Psi_0\rangle$ produces yet another universal solution, at least in the first order in deformation parameters $\varepsilon_n$. The effect of the operator appears as a change in the function $F(z)$,

$$|\tilde{\Psi}_0\rangle = e^{K(\varepsilon)}|\Psi_0\rangle \;\sim\; Q_L(\tilde{F})|I\rangle + C_L(\tilde{G})|I\rangle, \tag{3.14}$$

$$\tilde{F}(w) \;\equiv\; F(w) - \sum_{n=1}^{\infty} \varepsilon_n u_n(w)\partial F(w), \tag{3.15}$$

$$u_n(w) \equiv w(w^n - (-)^n w^{-n}). \tag{3.16}$$

It is easily confirmed that the deformed function also satisfy two conditions for a classical solution.[4]

The above calculation shows that we may explore the submanifold of classical solutions by the action of $e^{K(v)}$. This opens a possibility to relate different solutions written in the same form as described in (3.12). In particular, the series of solution constructed in [5, 6] may be related by the operator $e^{K(v)}$ with its parameters appropriately chosen. Of course, this cannot be realized by infinitesimal transformations discussed here and we have to consider finite transformations.

We make another observation that supports this idea. Consider the SFT defined around the classical solution $\Psi_0(h_a^l)$ written as eq. (2.11) with the function in (2.15). After taking the limit of $a \to -\frac{1}{2}$, we have the action $S[\Phi, Q_B^{(l)}]$ with the BRS charge given as

$$\begin{aligned} Q_B^{(l)} &= \frac{1}{2}Q_B + \frac{(-)^l}{4}(Q_{2l} + Q_{-2l}) + 2l^2\Big(c_0 - \frac{(-)^l}{2}(c_{2l} + c_{-2l})\Big) \\ &= Q(F^{(l)}) + C(G^{(l)}), \end{aligned} \tag{3.17}$$

[4] In eq. (3.15), we observe that no choice for a set of parameters leaves the function invariant. This implies that the symmetry generated by $K_n$ does not survive, at least, in its original form.

where the moments of the BRS current are defined in the expansion, $J_B(w) \equiv \sum_n Q_n w^{-n-1}$, and the functions $F^{(l)}(w)$ and $G^{(l)}(w)$ are given as

$$F^{(l)}(w) \equiv \frac{(-)^l}{4}\Big(w^l + (-w)^{-l}\Big)^2, \quad G^{(l)}(w) \equiv -l^2 w^{-2}(-)^l\Big(w^l - (-w)^{-l}\Big)^2. \tag{3.18}$$

If the classical solutions labeled by $l$ and $m$ ($l \neq m$) are really related by the finite form of the $K_n$ symmetry, the BRS charges in the actions $S[\Phi, Q_B^l]$ and $S[\Phi, Q_B^m]$ are to be related as

$$Q_B^{(m)} = e^{-K(v)} Q_{\rm B}^{(l)} e^{K(v)}, \tag{3.19}$$

with some operator $e^{K(v)}$, as we stated in (3.9). From eq. (3.17) and the commutation relation, $[K_n,\ Q_{2l}] = -2l(Q_{2l+n} - (-)^n Q_{2l-n})$, we realize that $K_n$ with $n =$ even are to be used for the purpose. So it is possible for eq. (3.19) to hold. We are to find out whether we may choose proper parameters so that eq. (3.19) holds. Further discussion of the transformation (3.19) will be given in the next section.

Before closing this section, we introduce operators which play an important role in relating BRS charges:

$$U_{2l}(t) = \exp\Big[-\frac{(-)^l}{4l}\ln\Big(\frac{1-t}{1+t}\Big)K_{2l}\Big] \qquad (l = 1, 2, ...). \tag{3.20}$$

When ordered with respect to the Virasoro operators, they are expressed as

$$U_{2l}(t) = \exp\Big(\frac{-(-)^l t}{2l} \cdot L_{-2l}\Big) \exp\Big(\frac{1}{2l}\ln(1-t^2) \cdot L_0\Big) \exp\Big(\frac{(-)^l t}{2l} \cdot L_{2l}\Big). \tag{3.21}$$

The operators in eq. (3.20) generate the conformal transformations [12],

$$f_{2l}(z, t) = \Big(\frac{z^{2l} - (-)^l\ t}{1 - (-)^l\ t z^{2l}}\Big)^{\frac{1}{2l}}. \tag{3.22}$$

A comment is in order. These conformal transformations are used earlier [8] in connection with the TTK solutions. The parameter $t$ introduced here corresponds to $Z(a) \equiv (1+a-\sqrt{1+2a})/a$ in the earlier expression.

# 4 Relating classical solutions with different values of $l$

When two classical solutions are related by the global transformation, expansions around them produce BRS charges satisfying eq. (3.9). Here we present a way to construct operators relating the BRS charges for the TTK classical solutions.

In the last section, we have seen that the BRS charges may be related as eq. (3.19) with the operator $K(v)$ written in terms of $K_n$ ($n =$even). Here we construct the operator $e^{K(v)}$ that

relates charges associated with $l = 1$ and $l \neq 1$. In the next subsection, it will be shown that the operator is realized in the limit of a one-parameter family of operators, $U(t)$ $(-1 \leq t \leq 0)$:

$$U(t=0) = 1, \qquad Q_B^{(l=1)} = \lim_{t \to -1} U(t) Q_B^{(l)} U^{-1}(t). \tag{4.1}$$

Unfortunately, there seems to be a subtlety in this operator: when trying to obtain the higher $l$ BRS charge from the $l = 1$ charge, we encounter a problem. This will be explained briefly in the second subsection. It is not clear to us at this moment whether this is just a technical problem or much deeper one. In the third subsection, the properties of the operator $U(t)$ are investigated. In particular, it will be shown that it has the well-defined normal ordered expression in terms of the Virasoro generators even in the limit of $t \to -1$.

## 4.1 Higher $l$ solutions down to $l = 1$ solution

Our construction of operators that relate BRS charges is based on the observation to be explained below. On the $l$-th BRS charge (3.17), we act the operator introduced in eq. (3.20) and find the charge transformed as,

$$U_{2l}(t) Q_B^{(l)} U_{2l}^{-1}(t) = Q(F_1^{(l)}) + C(G_1^{(l)}), \tag{4.2}$$

$$\begin{aligned} F_1^{(l)}(w,t) &\equiv F^{(l)}(z)|_{z=f_{2l}(w,-t)} = \frac{(-)^l}{4}\Big(w^l + (-w)^{-l}\Big)^2 \frac{(1+t)^2}{(1+(-)^l t\ w^{2l})\,(1+(-)^l t\ w^{-2l})} \\ &= \frac{1+t}{2} + \frac{1-t^2}{4}\sum_{n=1}^{\infty}(-)^{n-1}\ (-)^{ln} t^{n-1}(w^{2ln} + w^{-2ln}), \end{aligned} \tag{4.3}$$

$$\begin{aligned} G_1^{(l)}(w,t) &\equiv G^{(l)}(z)|_{z=f_{2l}(w,-t)} \times \Big(\frac{df_{2l}(w,-t)}{dw}\Big)^2 \\ &= -(-)^l l^2 w^{-2}\Big(w^l - (-w)^{-l}\Big)^2 \frac{(1+t)^2(1-t)^4}{(1+(-)^l t\ w^{2l})^3\,(1+(-)^l t\ w^{-2l})^3} \\ &= \frac{2l^2(1+t+t^2)}{w^2(1-t^2)} - \frac{l^2}{2w^2(1-t^2)}\sum_{n=1}^{\infty}(-)^{ln}\ g_n(t)(w^{2ln} + w^{-2ln}). \end{aligned} \tag{4.4}$$

Here $f_{2l}(w,t)$ is given in eq. (3.22) and the coefficients $g_n(t)$ are given by

$$g_n(t) = (-t)^{n-1}\Big[n^2 + n + 4(n+1)t - 2(n^2-2)t^2 - 4(n-1)t^3 + (n^2-n)t^4\Big]. \tag{4.5}$$

Substituting eqs. (4.3) and (4.4) into the terms on the rhs of eq. (4.2), we obtain the expression of $U_{2l}(t) Q_B^{(l)} U_{2l}^{-1}(t)$ as

$$\begin{aligned} U_{2l}(t) Q_B^{(l)} U_{2l}^{-1}(t) &= \frac{1+t}{2} Q_B + \frac{1-t^2}{4}\sum_{n=1}^{\infty}(-1)^{n(l-1)-1} t^{n-1}(Q_{2ln} + Q_{-2ln}) \\ &\quad + \frac{2(1+t+t^2)l^2}{1-t^2} c_0 - \frac{l^2}{2(1-t^2)}\sum_{n=1}^{\infty}(-)^{nl} g_n(t)(c_{2nl} + c_{-2ln}). \end{aligned} \tag{4.6}$$

On the rhs, we have higher moments of the BRS currents and pure ghost terms.

Note that, in the limit of $t \to -1$, all the terms containing the moments of BRS currents vanish. The remaining pure ghost terms become divergent. So some care should be taken when we take this limit at this stage. It would be worth pointing out that the pure ghost terms are indeed those appeared in the vacuum string field theory. This fact was first realized in Refs. [15, 17] and utilized recently to regularize the VSFT [18].

Another important observation on eq. (4.6) is the fact that the rhs contains the BRS charge itself. Since all the TTK solutions have this property, we may relate various BRS charges via the expression on the rhs of (4.6). In concrete, we act $U_2(t)$ for $l = 1$ on the rhs of (4.6) and see what would come out. Our expectation is that the result is somewhat close to the BRS charge for the $l = 1$ solution.

The action of $U_2(t)$, on the rhs of (4.6), replaces the argument of $F_1^{(l)}(w,t)$ by $f_2(w,t)$. After some calculations, we obtain

$$\begin{aligned} F_2^{(l)}(w,t) &\equiv F_1^{(l)}(z,t)|_{z=f_2(w,t)} \\ &= -\frac{1}{4}\Big(w-1/w\Big)^2\Big(\frac{2}{l+1}\Big)^2\frac{1}{\Big(1+\frac{l-1}{l+1}w^2\Big)\Big(1+\frac{l-1}{l+1}w^{-2}\Big)} + O\Big((1+t)\Big). \end{aligned} \tag{4.7}$$

Similarly we obtain the expression for $G_2^{(l)}(w,t)$. Note that the singular behavior in the limit of $t \to -1$, observed in the rhs expression of eq. (4.6), is absent in (4.7). The action of $U_2(t)$ has canceled the singular behavior. In the limit both $F_2^{(l)}(w,t)$ and $G_2^{(l)}(w,t)$ are finite. So we may take the limit of $t \to -1$ in the expression for $U_2(t)^{-1}U_{2l}(t)Q_B^{(l)}U_{2l}(t)^{-1}U_2(t)$ where $U(t) \equiv U_2(t)^{-1}U_{2l}(t)$:

$$\lim_{t\to-1} U_2(t)^{-1}U_{2l}(t)Q_B^{(l)}U_{2l}(t)^{-1}U_2(t) \equiv Q(\tilde{F}_2^{(l)}) + C(\tilde{G}_2^{(l)}) \ , \tag{4.8}$$

$$\tilde{F}_2^{(l)}(w) \equiv \lim_{t\to-1} F_2^{(l)}(w,t) = -\frac{1}{4}\Big(w-1/w\Big)^2\Big(\frac{2}{l+1}\Big)^2\frac{1}{\Big(1+\frac{l-1}{l+1}w^2\Big)\Big(1+\frac{l-1}{l+1}w^{-2}\Big)} \ , \tag{4.9}$$

$$\tilde{G}_2^{(l)}(w) \equiv -\frac{(\partial\tilde{F}_2^{(l)}(w))^2}{\tilde{F}_2^{(l)}(w)} \ .$$

Here we find the function $(w - 1/w)^2/4$ in eq. (4.9), the function for the $l = 1$ BRS charge. The remaining factor appeared in eq. (4.9) may be removed by an appropriate transformation. Indeed, we can deform the $l = 1$ BRS charge to the form of eq. (4.9):

$$U_2\Big(-\frac{l-1}{l+1}\Big)Q_B^{(1)}U_2\Big(-\frac{l-1}{l+1}\Big)^{-1} = Q(\tilde{F}_2^{(l)}) + C(\tilde{G}_2^{(l)}) \ . \tag{4.10}$$

Combining eqs. (4.8) and (4.10), we finally reach the $l=1$ BRS charge starting from the higher $l$ charge:

$$U_2^{-1}\Big(-\frac{l-1}{l+1}\Big)\lim_{t\to -1}\Big(U_2(t)^{-1}U_{2l}(t)Q_B^{(l)}U_{2l}(t)^{-1}U_2(t)\Big)U_2\Big(-\frac{l-1}{l+1}\Big)=Q_B^{(1)}\ .$$

The expression is also rewritten as

$$\lim_{t\to -1}U(t)Q_B^{(l)}U^{-1}(t)=Q_B^{(1)}\ , \tag{4.11}$$

where $U(t)\equiv U_2(\tilde{t})^{-1}U_{2l}(t)$ with $\tilde{t}$ defined as

$$\tilde{t}\equiv\frac{-(l-1)+(l+1)t}{(l+1)-(l-1)t}\ . \tag{4.12}$$

The parameter in $U_2$ is now $\tilde{t}$ due to the extra action of $U_2(-\frac{l-1}{l+1})$ in eq. (4.10).

## 4.2 $l=1$ solution up to $l=2$ solution

A natural question is whether we can obtain the SFT action around the $l=2$ solution starting from that for $l=1$ solution. As a relation of BRS charges, our question may be formulated as follows: is it possible to find an operator $\mathcal{U}$ such that $\mathcal{U}^{-1}Q_B^{(1)}\mathcal{U}=Q_B^{(2)}$? In this subsection, we explain what we have understood in relation to this question.

Let us reconsider eq. (4.11) for $l=2$. Before taking the limit, we write

$$Q_B^{(1)}(t)\equiv U_2^{-1}(\tilde{t})U_4(t)Q_B^{(2)}U_4^{-1}(t)U_2(\tilde{t})\equiv Q(F_t)+C(G_t) \tag{4.13}$$

where $\tilde{t}=(3t-1)/(3-t)$ as defined in eq. (4.12), and the function $F_t(w)$ is given as

$$F_t(w)=\frac{1}{4}(f^2+f^{-2})^2,\quad f(w;t)\equiv f_4\Big(f_2(w;\tilde{t});-t\Big). \tag{4.14}$$

Note here that $F_t(w)$ is slightly different from $F_2^{(l=2)}(w,t)$ since the former now includes the contribution from $U_2(-\frac{l-1}{l+1})|_{l=2}$ in eq. (4.10). Let us write the function $F_t(w)$ explicitly,

$$F_t(w)=\frac{1}{4}\cdot\frac{\Big(4\tilde{t}+(1+\tilde{t}^2)(w^2+w^{-2})\Big)^2}{\left(\frac{9t^2-14t+9}{(t-3)^2}+2\tilde{t}w^2+\left(\frac{t+1}{t-3}\right)^2w^4\right)\Big(w\to w^{-1}\Big)}. \tag{4.15}$$

In the limit of $t\to -1$, $F_t(w)$ becomes

$$F_t(w)\to -\frac{1}{4}(w-w^{-1})^2, \tag{4.16}$$

in other words,

$$Q_B^{(2)}(t) \to Q_B^{(1)},$$

the result of the previous subsection.

Now how about the inverse? The relation $U^{-1}(t)Q_B^{(1)}(t)U(t) = Q_B^{(2)}$ must hold since algebraically $U^{(-1)}(t)U(t) = 1$. Indeed, we will confirm this relation shortly. However, the relation is not the direct answer to our question. To find an answer, we would rather like to consider the operator $U^{-1}(t)Q_B^{(1)}(s)U(t)$. If we could take the limit of $s \to -1$ first, then take the other limit, the operator $\lim_{t\to -1} U(t)$ would be $\mathcal{U}$ we are looking for.

In calculating $U^{-1}(t)Q_B^{(1)}(s)U(t)$, we first consider $U_2(\tilde{t})Q_B^{(1)}(s)U_2^{-1}(\tilde{t})$. The function $F_s(w)$ in the integrand $Q(F_s)$ is replaced as[5]

$$F_s(w) \to F_s(z)|_{z=f_2(w;-\tilde{t})}. \tag{4.17}$$

An explicit calculation of the rhs of eq. (4.17) shows that the factors in the numerator and denominator of the resultant expression carries terms with $w^2$ and $w^4$ other than constants. In the next step, we let the operator $U_4(t)$ act on the rhs of eq. (4.17) and the variable $w$ is replaced by $f_4(w;t)$, which has the fourth order branch cuts. The factor $w^2$ is replaced by $\left(f_4(w;t)\right)^2$ and it still has branch cuts. Generically those produce the ambiguity in defining the contour integration. In order to avoid this difficulty, we require the terms of $w^2$ vanish on the rhs of eq. (4.17): this condition is found to be $s = t$. If we take $s = t$, there is no ambiguity and we find that $U^{-1}(t)Q_B^{(1)}(t)U(t)$ is certainly $Q_B^{(2)}$, even before taking the limit of $t \to -1$.

We have not been able to make sense of taking the limits of $U^{-1}(t)Q_B^{(1)}(s)U(t)$ independently. So we still have not reached the complete understanding of the relations between various TTK solutions.

## 4.3 Operator $U(t)$ and string field transformation

Though with some limitation, we have constructed the operator that relates the $l$-th and the first solutions of TTK solutions in the limit of $t \to -1$. Here we describe some properties of the operator: the differential equation; some evidence that suggests the well-definedness of the operator. For concreteness, we consider the operator $U(t) \equiv U_2^{-1}(\tilde{t})U_4(t)$ relating the first and the second solutions. With the operator $U(t = -1)$, we are supposed to be able to relate

[5] The other function $G_s(w)$ is also changed accordingly.

the string fields defined around two solutions: $|\Phi^{(2)}\rangle = U(-1)^{-1}|\Phi^{(1)}\rangle$. Later, we will show the relation in terms of components fields.

In deriving the differential equation for $U(t)$, the parameter $t$ is to be restricted for $|t| < 1$ as we will see shortly. It is straightforward to obtain the equation

$$\frac{d}{dt}U(t) = \frac{1}{2}\Big(K_2 + \frac{1}{2}U_2(-\tilde{t})K_4U_2^{-1}(-\tilde{t})\Big)\frac{U(t)}{1-t^2}. \tag{4.18}$$

The operator $U_2(-\tilde{t})K_4U_2^{-1}(-\tilde{t})$ may further be calculated as

$$U_2(-\tilde{t})K_4U_2^{-1}(-\tilde{t}) = \oint \frac{dz}{2\pi i} z(z^4 - z^{-4})\Big(\frac{df_2(z;-\tilde{t})}{dz}\Big)^2 T\Big(f_2(z;-\tilde{t})\Big). \tag{4.19}$$

where the integration path is over the unit circle. By changing the variable from $z$ to $u \equiv f_2(z;-\tilde{t})$, we obtain the expression of the operator

$$U_2(-\tilde{t})K_4U_2^{-1}(-\tilde{t}) = \oint \frac{du}{2\pi i} u \frac{(1+\tilde{t}^2)(u^4 - u^{-4}) + 4\tilde{t}(u^2 - u^{-2})}{(1+\tilde{t}^2u^2)(1+\tilde{t}^2u^{-2})} T(u). \tag{4.20}$$

It is easy to understand the integration path for $u$ is again over the unit circle, when the parameter $t$ is real and $|t| < 1$. When we take $t = -1$, the integration path is not clearly defined. The value is to be reached only in the limiting procedure. The integrand of (4.20) may be expanded with respect to $\tilde{t}$ since $|\tilde{t}| < 1$ for $|t| < 1$. We finally obtain the differential equation for $U(t)$,

$$\begin{aligned} \frac{d}{dt}U(t) &= K(t)U(t), \\ K(t) &\equiv \frac{(7-5t)(1+t)}{(1-t)(3-t)^3}K_2 + \frac{16(1-t^2)}{(t-3)^4}\sum_2^\infty\Big(\frac{1-3t}{3-t}\Big)^{n-2}K_{2n}. \end{aligned} \tag{4.21}$$

Now, we show that $U(t)^{-1}$ has a well-defined normal ordered expression:

$$U(t)^{-1} = \exp\left(\sum_{n=1}^\infty v_{-2n}L_{-2n}\right)\exp\left(v_0L_0\right)\exp\left(\sum_{n=1}^\infty v_{2n}L_{2n}\right) \tag{4.22}$$

where the coefficients $v_n = v_n(t)$ are finite even in the limit of $t \to -1$. The normal ordered form in eq. (4.22) are expressed with even modes Virasoro operators since $U_2(t)$ and $U_{2l}(t)$ themselves are written with $L_{\pm 2}$ and $L_{\pm 2l}$ and the algebra of even mode operators is closed.

In the following, we take the operator $U(t)$ relating the solutions $l = 1$ and $2$ as an example and confirm our claim. We have not noticed any difficulty to extend our analysis to other cases.

Taking $|h\rangle$, a highest weight state with the dimension $h$, we calculate various matrix elements of $U(t)^{-1}$ in terms of $v_n(t)$ :

$$\langle h|\, U(t)^{-1}\, |h\rangle \;=\; e^{v_0 h}, \tag{4.23}$$

$$\langle h|\, U(t)^{-1}\, L_{-2}\, |h\rangle \;=\; 4h\, v_2\, e^{v_0 h}, \tag{4.24}$$

$$\langle h|\, L_2\, U(t)^{-1}\, |h\rangle \;=\; 4h\, v_{-2}\, e^{v_0 h}, \tag{4.25}$$

$$\langle h|\, L_4\, U(t)^{-1}\, |h\rangle \;=\; \left\{12h\,(v_{-2})^2 + 8h\, v_{-4}\right\} e^{v_0 h}, \tag{4.26}$$

$$\langle h|\, U(t)^{-1}\, (L_{-2})^2\, |h\rangle \;=\; \left\{16h\,(v_2)^2 + 24h\, v_4\right\} e^{v_0 h}, \tag{4.27}$$

and so on. Then, if all of these matrix elements are obtained, we can determine the coefficients $v_n(t)$ iteratively. In appendix C, we calculate some of the matrix elements using the definition of $U(t)^{-1}$. As a result, the coefficients $v_n(t)$ are found to be

$$v_0(t) \;=\; \frac{1}{4}\ln\left\{\frac{64(1-t)^3(3-t)^2}{(9t^2-14t+9)^3}\right\}, \tag{4.28}$$

$$v_2(t) \;=\; \frac{(6t^2+4t-18)(3t-1)}{4(9t^2-14t+9)(3-t)}, \tag{4.29}$$

$$v_{-2}(t) \;=\; \frac{3t-1}{2}\left(\frac{1-t}{9t^2-14t+9}\right)^{\frac{1}{2}}, \tag{4.30}$$

$$v_4(t) \;=\; \frac{8t(1-t)^2(9t^3-4t^2-11t+18)}{(t-3)^2(9t^2-14t+9)^2}, \tag{4.31}$$

$$v_{-4}(t) \;=\; \frac{t}{4}. \tag{4.32}$$

We should emphasize that there is no singularity in $v_n(t)$ in the limit $t \to -1$. Finally, taking the limit, we can obtain the normal ordered expression of $U(-1)^{-1}$:

$$\begin{aligned} &U(-1)^{-1} = \lim_{t\to -1} U(t)^{-1} \\ =\;& \exp\left(-\frac{1}{2}L_{-2} - \frac{1}{4}L_{-4} + \cdots\right) \exp\left(-\frac{1}{2}\log 2\, L_0\right) \exp\left(\frac{1}{8}L_2 + \frac{1}{32}L_4 + \cdots\right). \end{aligned} \tag{4.33}$$

Now, let us consider the string field transformation $|\Phi^{(2)}\rangle = U(-1)^{-1}|\Phi^{(1)}\rangle$, by which two theories expanded around $l=1$ and $l=2$ solutions can be related. Since the operator $U(-1)^{-1}$ has the normal ordered expression (4.33), the string field transformation has a well-defined Fock space expression, namely we can obtain transformations for all component fields without any divergence.

Write the string field up to level two as

$$|\Phi^{(1)}\rangle \;=\; \phi(x)\, c_1\, |0\rangle$$

$$
\begin{aligned}
&+A_\mu(x)\,c_1\alpha_{-1}^\mu\,|0\rangle + iB(x)\,c_0\,|0\rangle \\
&+\psi_{\mu\nu}(x)\,c_1\alpha_{-1}^\mu\alpha_{-1}^\nu\,|0\rangle + ia_\mu(x)\,c_1\alpha_{-2}^\mu\,|0\rangle \\
&+s(x)\,c_{-1}\,|0\rangle + t(x)\,c_0c_1b_{-2}\,|0\rangle + iu_\mu(x)\,c_0\alpha_{-1}^\mu\,|0\rangle + \cdots .
\end{aligned}
\tag{4.34}
$$

Acting the normal ordered expression (4.33) of $U(-1)^{-1}$ on the string field (4.34),[6] we can easily find transformations for these component fields:

$$
\begin{aligned}
\phi'(x) &= \sqrt{2}e^{\frac{\alpha'}{2}\log 2\cdot\partial^2}\left(\phi(x) + \frac{1}{8}\psi_\mu^\mu(x) + \frac{\sqrt{2\alpha'}}{4}\partial_\mu a^\mu(x) - \frac{3}{8}s(x) + \frac{1}{2}t(x)\cdots\right), \\
A'_\mu(x) &= e^{\frac{\alpha'}{2}\log 2\cdot\partial^2}\left(A_\mu(x) + \cdots\right), \\
B'(x) &= e^{\frac{\alpha'}{2}\log 2\cdot\partial^2}\left(B(x) + \cdots\right), \\
\psi'_{\mu\nu}(x) &= e^{\frac{\alpha'}{2}\log 2\cdot\partial^2}\left\{-\frac{\sqrt{2}}{4}g_{\mu\nu}\,\phi(x) + \frac{1}{\sqrt{2}}\psi_{\mu\nu}(x)\right. \\
&\qquad \left. -\frac{\sqrt{2}}{4}g_{\mu\nu}\left(\frac{1}{8}\psi_\rho^\rho(x) + \frac{\sqrt{2\alpha'}}{4}\partial_\rho a^\rho(x) - \frac{3}{8}s(x) + \frac{1}{2}t(x)\right) + \cdots\right\}, \\
a'_\mu(x) &= e^{\frac{\alpha'}{2}\log 2\cdot\partial^2}\left\{\sqrt{\alpha'}\partial_\mu\phi(x) + \frac{1}{\sqrt{2}}a_\mu(x)\right. \\
&\qquad \left. +\sqrt{\alpha'}\partial_\mu\left(\frac{1}{8}\psi_\rho^\rho(x) + \frac{\sqrt{2\alpha'}}{4}\partial_\rho a^\rho(x) - \frac{3}{8}s(x) + \frac{1}{2}t(x)\right) + \cdots\right\}, \\
s'(x) &= e^{\frac{\alpha'}{2}\log 2\cdot\partial^2}\left\{-\frac{3\sqrt{2}}{2}\phi(x) + \frac{1}{\sqrt{2}}s(x)\right. \\
&\qquad \left. -\frac{3\sqrt{2}}{2}\left(\frac{1}{8}\psi_\rho^\rho(x) + \frac{\sqrt{2\alpha'}}{4}\partial_\rho a^\rho(x) - \frac{3}{8}s(x) + \frac{1}{2}t(x)\right) + \cdots\right\}, \\
t'(x) &= e^{\frac{\alpha'}{2}\log 2\cdot\partial^2}\left\{\sqrt{2}\phi(x) + \frac{1}{\sqrt{2}}t(x)\right. \\
&\qquad \left. +\sqrt{2}\left(\frac{1}{8}\psi_\rho^\rho(x) + \frac{\sqrt{2\alpha'}}{4}\partial_\rho a^\rho(x) - \frac{3}{8}s(x) + \frac{1}{2}t(x)\right) + \cdots\right\}, \\
u'_\mu(x) &= e^{\frac{\alpha'}{2}\log 2\cdot\partial^2}\left(\frac{1}{\sqrt{2}}u_\mu(x) + \cdots\right),
\end{aligned}
$$

where the abbreviation denotes contributions from the higher level component fields.

The string field transformation has a well-defined expression. It mixes tensor fields of various ranks; On each component, it is a non-local transformation due to infinite derivative terms.

[6] We have used the commutation relations $[L_m,\,\alpha_n^\mu] = -n\alpha_{m+n}^\mu$, $[L_m,\,c_n] = -(2m+n)c_{m+n}$, $[L_m,\,b_n] = (m-n)b_{m+n}$, and $[L_m,\,\varphi(x)] = -i\sqrt{2\alpha'}\partial_\mu\varphi(x)\alpha_m^\mu \quad (m \neq 0)$ for any component fields $\varphi(x)$.

# 5 Discussion

In this paper, we addressed the question how the presence of many analytic classical solutions for SFT could be consistent with the physical picture of the tachyon condensation. Our result suggests that they are related by a particular type of gauge transformations. In more concrete terms, we have seen that the BRS charge for the $l$-th classical solution ($l \neq 1$) can be transformed down to that for $l = 1$. The inverse operation has some subtlety as explained in section 4. The transformation is generated by operators $K_n$ ($n =$ even). We observed that the symmetry generated by operators $K_n$ are to be regarded as the "global" part of the SFT gauge symmetry. The situation is summarized in the following sequence,

$$\begin{aligned}
&\{\text{The stringy gauge symmetry} : \Psi' = U^{-1} * Q_{\rm B} U + U^{-1} * \Psi * U\} \\
&\supset \{\text{Its global subset} : \Psi' = U^{-1} * \Psi * U \text{ with } Q_{\rm B} U = 0\} \\
&\supset \{\text{The symmetry generated with } K_n : \Psi' = \exp(K(v))\Psi\} \\
&\supset \{\text{The symmetry generated with } K_n \ (n = \text{even})\}.
\end{aligned}$$

In relating TTK solutions, we have utilized the last subset in the above sequence. Generically speaking, solutions are to be related by the gauge symmetry. So our approach may be too restrictive and that could be the reason why we encounter the subtlety.

In order to confirm that the operator relating BRS charges is well-defined, we studied properties of the operator that transforms $l = 2$ BRS charge into $l = 1$ charge and found that it has a well-defined normal ordered expression in terms of the Virasoro generators.

We studied relations between solutions obtained in Refs. [5, 6]. Another important direction of investigation is to find how those solutions could be related to solutions obtained in different approaches, eg, the level truncation [3, 19, 20, 21].

Most of the works on classical solutions for CSFT have been performed in the Siegel gauge. However, the universal solution proposed by [5] cannot be in the Siegel gauge as explained in the appendix D. A transformation generated by $K_n$ cannot bring a universal solution into the Siegel gauge: we have to consider more general gauge transformation.

In relation to the VSFT conjecture and the TTK solutions, recently there appeared an interesting paper [18]. The VSFT conjecture on the tachyon vacuum implies that the action expanded around a TTK solution must be related to VSFT via an appropriate transformation of the string field. The authors of [18] discussed this possibility and constructed, with the level truncation technique and a regulated butterfly state, a classical solution that could clarify this point.

Before closing, let us add a few remarks. 1) The cohomology analysis around a classical solution has shown that the ghost numbers of non-trivial states depend on the value of $l$ [6, 7]. It would be interesting to see how operators, eg, $U(t)$ in section 4, relate cohomologically non-trivial states obtained for various $l$. That would be another non-trivial test of those operators and the question certainly deserves further study. 2) We wonder what happens to the symmetries of the SFT defined around the non-trivial classical solution. Here we make an observation that symmetries generated by $K_n$ are broken on these classical solutions. When we consider an infinitesimal change of the $l$-th solution in Ref. [6], the function $F_{2l}(w)$ is transformed as shown in eq. (3.15). It is easy to see that any choice of the parameters $\varepsilon_n$ do not leave the function invariant. This implies the breaking of the symmetries: the symmetry generated by $K_n$ does not survive the tachyon condensation, at least, in its original form. 3) The function $F(z)$ in the generic form of classical solution (3.12) is to satisfy two conditions $F(-1/w) = F(w)$ and $F(\pm i) = 0$ and is related to another function $G(z)$ as in eq. (3.13). Strictly speaking, in order for (3.12) to be a classical solution, these conditions are required to hold only on the unit circle. We encounter the same situation in eq. (B.8).

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# A Evaluating $S[g^{-1} * Q_B g, Q_B]$ for $g$ connected to $I$

We consider the stringy gauge functional $g$ that may be continuously deformed to the identity $I$: ie, we assume that there exit a one-parameter family of functionals $g(t)$ $(0 \leq t \leq 1)$ so that $g(0) = I$ and $g(t = 1) = g$. Construct the pure gauge string field as

$$\Psi(t) \equiv g^{-1}(t) * Q_B \; g(t). \tag{A.1}$$

By using the properties (4), it is easy to show that the string field $\Psi(t)$ satisfies the equation of motion (2.4). Note also that $\Psi(t = 0) = 0$. Now we may calculate the variation of the

action for $\Psi(t)$ as

$$\frac{d}{dt}S[\Psi(t), Q_B] = \frac{1}{g^2}\int\Big(Q_B\Psi(t) + \Psi(t)*\Psi(t)\Big)*\frac{d}{dt}\Psi(t)\ . \tag{A.2}$$

The rhs of (A.2) is proportional to the equation of motion, and it vanishes. Obviously, it holds that $S[\Psi(t=0), Q_B] = 0$. Therefore $S[\Psi(t), Q_B] = 0$ for any value of $t$, in particular $t = 1$ [26].

# B On the global transformation given in eq. (3.5)

The energy momentum tensor $T(w)$ is expanded by the Virasoro operator $L_n$ as

$$T(w) = \sum_{n=-\infty}^{\infty} L_n w^{-n-1}. \tag{B.1}$$

From commutation relations of $L_n$, we can derive the commutation relation between $T(w)$ and $T(w')$ as

$$[T(w),\, T(w')] = -\partial T(w)\delta(w,w') + T(w)\partial_{w'}\delta(w,w'), \tag{B.2}$$

where the delta function is defined by $\delta(w,w') = \sum_n w^{-n}w'^{n-1}$. Here, we define half string operators associated with the energy-momentum tensor as follows,

$$\mathcal{T}_L(f) = \int_{C_{\rm left}}\frac{dw}{2\pi i}f(w)T(w), \quad \mathcal{T}_R(f) = \int_{C_{\rm right}}\frac{dw}{2\pi i}f(w)T(w). \tag{B.3}$$

Using (B.2) and the splitting properties of the delta function [5], we can find the commutation relations between these operators:

$$[\mathcal{T}_L(f),\, \mathcal{T}_L(g)] = \mathcal{T}_L((\partial f)g - f\partial g), \tag{B.4}$$

$$[\mathcal{T}_R(f),\, \mathcal{T}_R(g)] = \mathcal{T}_R((\partial f)g - f\partial g), \tag{B.5}$$

$$[\mathcal{T}_L(f),\, \mathcal{T}_R(g)] = 0, \tag{B.6}$$

where the functions $f(w)$ and $g(w)$ satisfy $f(\pm i) = g(\pm i) = 0$.

If the function $f(w)$ satisfies $f(w) = (dw/d\tilde{w})f(\tilde{w})$ for $\tilde{w} = -1/w$, we find that

$$dw f(w)T(w) = d\tilde{w} f(\tilde{w})\tilde{T}(\tilde{w}) \tag{B.7}$$

since $T(w)$ is a primary field with the conformal dimension 2 for $c = 0$ [22]. Using the relation (B.7), we can obtain two properties of the half string operators for $f(w)$ such that $f(w) = (dw/d\tilde{w})f(\tilde{w})$:

$$\mathcal{T}_R(f)A * B = -A * \mathcal{T}_L(f)B, \quad \mathcal{T}_R(f)I + \mathcal{T}_L(f)I = 0, \tag{B.8}$$

where $A$ and $B$ denote arbitrary string fields and $I$ is the identity string field. In deriving eq. (B.8), it is suffice for the function $f(w)$ to satisfy the condition, $f(w) = (dw/d\tilde{w})f(\tilde{w})$ for $\tilde{w} = -1/w$, on the unit circle.

We consider gauge transformation with the string functional

$$g = \exp\left(-\mathcal{T}_L(f)I\right), \tag{B.9}$$

where $f(w)$ is required to satisfy the condition $f(w) = (dw/d\tilde{w})f(\tilde{w})$. It is easy to see that the other condition $f(\pm i) = 0$ follows from the former. The gauge transformation is given by

$$\Psi' = g^{-1} * Q_{\mathrm{B}} g + g^{-1} * \Psi * g. \tag{B.10}$$

The first term becomes zero since $[Q_{\mathrm{B}},\, \mathcal{T}_L(f)] = 0$ and $Q_{\mathrm{B}} I = 0$. Then, from the equations (B.8), the gauge transformation can be rewritten as the string field redefinition

$$\Psi' = \exp(\mathcal{T}(f))\,\Psi, \tag{B.11}$$

where the operator $\mathcal{T}(f)$ is defined as $\mathcal{T}(f) = \mathcal{T}_L(f) + \mathcal{T}_R(f)$.

# C Matrix elements of $U(t)^{-1}$

The operators $U_2(\tilde{t})$ and $U_4(t)^{-1}$ can be written using normal ordered expression as

$$U_2(\tilde{t}) = e^{\frac{\tilde{t}}{2}L_{-2}}\, e^{\frac{1}{2}\log(1-\tilde{t}^2)\,L_0}\, e^{-\frac{\tilde{t}}{2}L_2}, \quad U_4(t)^{-1} = e^{\frac{t}{4}L_{-4}}\, e^{\frac{1}{4}\log(1-t^2)\,L_0}\, e^{-\frac{t}{4}L_4}. \tag{C.1}$$

Take a normalized highest weight state with the dimension $h$, $|h\rangle$, and calculate the matrix element $\langle h|\, U(t)^{-1}\, |h\rangle$,

$$\langle h|\, U(t)^{-1}\, |h\rangle = \langle h|\, U_4(t)^{-1} U_2(\tilde{t})\, |h\rangle = (1-t^2)^{\frac{h}{4}}(1-\tilde{t}^2)^{\frac{h}{2}}\, \langle h|\, e^{-\frac{t}{4}L_4}\, e^{\frac{\tilde{t}}{2}L_{-2}}\, |h\rangle\ . \tag{C.2}$$

We can derive the recursion relation for matrix elements $\langle h|\,(L_4)^n (L_{-2})^{2n}\,|h\rangle$,

$$\langle h|\,(L_4)^n(L_{-2})^{2n}\,|h\rangle = 8n(2n-1)(4n+3h-4)\,\langle h|\,(L_4)^{n-1}(L_{-2})^{2(n-1)}\,|h\rangle\,, \tag{C.3}$$

which can be solved to give the expression

$$\langle h|\,(L_4)^n(L_{-2})^{2n}\,|h\rangle = 64^n n! \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(n+\frac{3h}{4}\right)}{\Gamma\left(\frac{3h}{4}\right)}. \tag{C.4}$$

Using eq. (C.4), we obtain

$$\langle h|\, e^{-\frac{t}{4}L_4}\, e^{\frac{\tilde{t}}{2}L_{-2}}\, |h\rangle = (1+t\tilde{t}^2)^{-\frac{3h}{4}}. \tag{C.5}$$

Substituting (C.5) into (C.2), we reach the final expression for $\langle h| U(-1)^{-1} |h\rangle$:

$$\langle h| U(t) |h\rangle = \left\{ \frac{64(1-t)^3(3-t)^2)}{(9t^2-14t+9)^3} \right\}^{\frac{h}{4}} . \tag{C.6}$$

Next, let us calculate the matrix element $\langle h| U(t)^{-1} L_{-2} |h\rangle$.

$$\begin{aligned} \langle h| U(t)^{-1} L_{-2} |h\rangle &= (1-t^2)^{\frac{h}{4}} (1-\tilde{t}^2)^{\frac{h+2}{2}} \langle h| e^{-\frac{t}{4}L_4} e^{\frac{\tilde{t}}{2}L_{-2}} L_{-2} |h\rangle \\ & \quad -2\, h\, \tilde{t}(1-t^2)^{\frac{h}{4}}(1-\tilde{t}^2)^{\frac{h}{2}} \langle h| e^{-\frac{t}{4}L_4} e^{\frac{\tilde{t}}{2}L_{-2}} |h\rangle . \end{aligned} \tag{C.7}$$

Differentiating eq. (C.5) with respect to $\tilde{t}$, we find

$$\langle h| e^{-\frac{t}{4}L_4} e^{\frac{\tilde{t}}{2}L_{-2}} L_{-2} |h\rangle = -3ht\tilde{t}(1+t\tilde{t}^2)^{-\frac{3h}{4}-1}. \tag{C.8}$$

Combining the results (C.5) and (C.8) with (C.7), we find

$$\begin{aligned} \langle h| U(t)^{-1} L_{-2} |h\rangle &= h\, \frac{\tilde{t}(t\tilde{t}^2-3t-2)}{1+t\tilde{t}^2} \left\{ \frac{(1-t^2)(1-\tilde{t}^2)^2}{(1+t\tilde{t}^2)^3} \right\}^{\frac{h}{4}} \\ &= h\, \frac{2(3t^2+2t-9)(3t-1)}{(9t^2-14t+9)(3-t)} \left\{ \frac{64(1-t)^3(3-t)^2)}{(9t^2-14t+9)^3} \right\}^{\frac{h}{4}} . \end{aligned} \tag{C.9}$$

Other matrix elements may be calculated in a similar manner. Using the normal ordered expression of $U_2(\tilde{t})$ and $U_4(t)$, we easily find

$$\begin{aligned} \langle h| L_2 U(t)^{-1} |h\rangle &= (1-t^2)^{\frac{h+2}{4}} (1-\tilde{t}^2)^{\frac{h}{2}} \\ & \quad \times \sum_{n=0}^{\infty} \frac{1}{(2n+1)!\, n!} \left(\frac{\tilde{t}}{2}\right)^{2n+1} \left(-\frac{t}{4}\right)^n \langle h| L_2\, (L_4)^n (L_{-2})^{2n+1} |h\rangle , \end{aligned} \tag{C.10}$$

$$\begin{aligned} \langle h| L_4 U(t)^{-1} |h\rangle &= (1-t^2)^{\frac{h+4}{4}} (1-\tilde{t}^2)^{\frac{h}{2}} \langle h| L_4\, e^{-\frac{t}{4}L_4} e^{\frac{\tilde{t}}{2}L_{-2}} |h\rangle \\ & \quad +(1-t^2)^{\frac{h}{4}}(1-\tilde{t}^2)^{\frac{h}{2}}\, 2h\, t\, \langle h| e^{-\frac{t}{4}L_4} e^{\frac{\tilde{t}}{2}L_{-2}} |h\rangle , \end{aligned} \tag{C.11}$$

$$\begin{aligned} \langle h| U(t)^{-1} (L_{-2})^2 |h\rangle &= (1-t^2)^{\frac{h}{4}} (1-\tilde{t}^2)^{\frac{h+4}{2}} \langle h| e^{-\frac{t}{4}L_4} e^{\frac{\tilde{t}}{2}L_{-2}} (L_{-2})^2 |h\rangle \\ & \quad -4(h+1)(1-t^2)^{\frac{h}{4}}\, \tilde{t}\, (1-\tilde{t}^2)^{\frac{h+2}{2}} \langle h| e^{-\frac{t}{4}L_4} e^{\frac{\tilde{t}}{2}L_{-2}}\, L_{-2} |h\rangle \\ & \quad +4h(h+1)(1-t^2)^{\frac{h}{4}} \tilde{t}^2 (1-\tilde{t}^2)^{\frac{h}{2}} \langle h| e^{-\frac{t}{4}L_4} e^{\frac{\tilde{t}}{2}L_{-2}} |h\rangle . \end{aligned} \tag{C.12}$$

We can calculate (C.10) by using

$$\langle h| L_2\, (L_4)^n (L_{-2})^{2n+1} |h\rangle = 4^{3n+1} h\, n!\, \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \frac{\Gamma\left(n+\frac{3h}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{3h}{4}+\frac{1}{2}\right)}, \tag{C.13}$$

and eqs. (C.11) and (C.12) can be evaluated by using eqs. (C.5), (C.8) and

$$\begin{aligned} \langle h| L_4\, e^{-\frac{t}{4}L_4} e^{\frac{\tilde{t}}{2}L_{-2}} |h\rangle &= 3h\, \tilde{t}^2\, (1+t\tilde{t}^2)^{-\frac{3h}{4}-1}, \\ \langle h| e^{-\frac{t}{4}L_4} e^{\frac{\tilde{t}}{2}L_{-2}} (L_{-2})^2 |h\rangle &= 3h\, t\, (1+t\tilde{t}^2)^{-\frac{3h}{4}-2}(-2-2\, t\, \tilde{t}^2+3h\, t\, \tilde{t}^2). \end{aligned}$$

Finally, we obtain the expressions for the matrix elements,

$$
\begin{aligned}
\langle h|\, L_2\, U(t)^{-1}\, |h\rangle &= 2h\,\tilde{t}\left(\frac{1-t^2}{1+t\tilde{t}^2}\right)^{\frac{1}{2}} \left\{\frac{(1-t^2)(1-\tilde{t}^2)^2}{(1+t\tilde{t}^2)^3}\right\}^{\frac{h}{4}} \\
&= 2h\,\frac{(3t-1)(1-t)^{\frac{1}{2}}}{(9t^2-14t+9)^{\frac{1}{2}}} \left\{\frac{64(1-t)^3(3-t)^2)}{(9t^2-14t+9)^3}\right\}^{\frac{h}{4}}, \\
\langle h|\, L_4\, U(t)^{-1}\, |h\rangle &= h\,\frac{2t+3\tilde{t}^2-t^2\tilde{t}^2}{1+t\tilde{t}^2} \left\{\frac{(1-t^2)(1-\tilde{t}^2)^2}{(1+t\tilde{t}^2)^3}\right\}^{\frac{h}{4}} \\
&= h\,\frac{-9t^3+17t^2-3t+3}{9t^2-14t+9} \left\{\frac{64(1-t)^3(3-t)^2)}{(9t^2-14t+9)^3}\right\}^{\frac{h}{4}}, \\
\langle h|\, U(t)^{-1}\,(L_{-2})^2\, |h\rangle &= \Bigg[16h(h+1)\left\{\frac{\tilde{t}(t\tilde{t}^2-3t-2)}{4(1+t\tilde{t}^2)}\right\}^2 \\
&\quad -24h\,\frac{t(1-\tilde{t}^2)(2+t\tilde{t}^2)}{8(1+t\tilde{t}^2)^2}\Bigg] \left\{\frac{(1-t^2)(1-\tilde{t}^2)^2}{(1+t\tilde{t}^2)^3}\right\}^{\frac{h}{4}} \\
&= \Bigg[16h(h+1)\left\{\frac{(3t-1)(3t^2+2t-9)}{2(3-t)(9t^2-14t+9)}\right\}^2 \\
&\quad -24h\,\frac{8t(1-t)^2(9t^3-4t^2-11t+18)}{(t-3)^2(9t^2-14t+9)^2}\Bigg] \left\{\frac{64(1-t)^3(3-t)^2)}{(9t^2-14t+9)^3}\right\}^{\frac{h}{4}}. \qquad \text{(C.14)}
\end{aligned}
$$

# D The universal solutions are not in the Siegel gauge

In this appendix, we show that the universal solution given in Ref. [5] does not satisfy the Siegel gauge condition.

The universal solution is written in the following form

$$
|\Psi_0\rangle = Q_{\mathrm{L}}(F)\,|I\rangle + C_{\mathrm{L}}(G)\,|I\rangle\,, \qquad \text{(D.1)}
$$

$$
G = -\frac{(\partial F)^2}{1+F}. \qquad \text{(D.2)}
$$

Let us search for the function $F(w)$ which satisfies the Siegel gauge condition

$$
0 = b_0\,|\Psi_0\rangle = \int_{C_{\mathrm{left}}}\frac{dw}{2\pi i}F(w)b_0J_{\mathrm{B}}(w)\,|I\rangle + \int_{C_{\mathrm{left}}}\frac{dw}{2\pi i}G(w)b_0c(w)\,|I\rangle\,, \qquad \text{(D.3)}
$$

relying on the conformal technique. First note that the identity state $|I\rangle$ can be written as $|I\rangle = U^{-1}_{I\circ h\circ I}\,|0\rangle$. That is, the state can be expressed with the operator for the conformal transformation,

$$
I\circ h\circ I = \frac{w^2-1}{2w} \equiv g(w).
$$

Using $Q_n |0\rangle = 0\,(n \geq 0)$ and

$$U_g J_{\rm B}(w) U_g^{-1} = [\partial g(w)]^{+1} J_{\rm B}(g(w)) = \partial g(w) \sum_{n=-\infty}^{\infty} (g(w))^{-n-1} Q_n,$$

we find

$$J_{\rm B}(w) |I\rangle = U_g^{-1} \sum_{n=1}^{\infty} \partial g(w) (g(w))^{n-1} Q_{-n} |0\rangle .$$

So we obtain the expression for the first term of eq. (D.1)

$$Q_{\rm L}(F) |I\rangle \;\; = \;\; U_g^{-1} \sum_{n=1}^{\infty} \alpha_n Q_{-n} |0\rangle \tag{D.4}$$

where $\alpha_n$ is given as

$$\alpha_n = \int_{\rm C_{left}} \frac{dw}{2\pi i} F(w) (g(w))^{n-1} \partial g(w). \tag{D.5}$$

We rewrite the second term $C_{\rm L}(G) |I\rangle$ in a similar manner: since

$$U_g c(w) U_g^{-1} = [\partial g(w)]^{-1} c(g(w)) = [\partial g(w)]^{-1} \sum_{n=-\infty}^{\infty} (g(w))^{-n+1} c_n,$$

we obtain

$$C_{\rm L}(G) |I\rangle = U_g^{-1} \sum_{n=-1}^{\infty} \beta_n c_{-n} |0\rangle , \tag{D.6}$$

with $\beta_n$ given as

$$\beta_n \equiv \int_{\rm C_{left}} \frac{dw}{2\pi i} G(w) (g(w))^{n+1} (\partial g(w))^{-1}. \tag{D.7}$$

From eqs. (D.4) and (D.6), the universal solution is now rewritten as

$$|\Psi_0\rangle = U_g^{-1} \left\{ \sum_{n=1}^{\infty} \alpha_n Q_{-n} |0\rangle + \sum_{n=-1}^{\infty} \beta_n c_{-n} |0\rangle \right\} . \tag{D.8}$$

Let us write the gauge condition for the universal solution written as eq. (D.8). First note

$$U_g b_0 U_g^{-1} = \oint \frac{dw}{2\pi i} w U_g b(w) U_g^{-1} = \oint \frac{dw}{2\pi i} w (\partial g(w))^2 b(g(w)) = \sum_{n=-\infty}^{\infty} \gamma_n b_{-n}, \tag{D.9}$$

where $\gamma_n$ is

$$\gamma_n = \oint \frac{dw}{2\pi i} w (\partial g(w))^2 (g(w))^{n-2} = \oint \frac{dw}{2\pi i} w \left( \frac{1+w^2}{2w^2} \right)^2 \left( \frac{w^2-1}{2w} \right)^{n-2} . \tag{D.10}$$

After some calculation, we find that $\gamma_n$ vanish for $n \leq -1$ and $n =$odd. Therefore the condition may be written

$$0 = b_0 \left|\Psi_0\right\rangle = U_g^{-1} \left\{ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \gamma_{2m} \alpha_n \cdot b_{-2m} Q_{-n} \left|0\right\rangle + \sum_{m=0}^{\infty} \sum_{n=-1}^{\infty} \gamma_{2m} \beta_n \cdot b_{-2m} c_{-n} \left|0\right\rangle \right\},$$

or,

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \gamma_{2m} \alpha_n \cdot b_{-2m} Q_{-n} \left|0\right\rangle + \sum_{m=0}^{\infty} \sum_{n=-1}^{\infty} \gamma_{2m} \beta_n \cdot b_{-2m} c_{-n} \left|0\right\rangle = 0. \tag{D.11}$$

Explicitly writing the state $b_{-2m} Q_{-n} \left|0\right\rangle$ $(m \geq 1, n \geq 1)$ as

$$b_{-2m} Q_{-n} \left|0\right\rangle = b_{-2m} c_0 L_{-n}^X \left|0\right\rangle + \cdots,$$

we realize that the first and second terms in eq. (D.11) cannot cancel with each other. Thus $\alpha_n = 0 \ \ (n \geq 1)$ as well as $\beta_n = 0 \ \ (n \geq -1)$. Rewriting (D.5) with $w = e^{i\sigma}$, we find

$$\alpha_n = i^{n-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\sigma}{2\pi} F(\sigma) \left(\sin\sigma\right)^{n-1} \cos\sigma = \int_{-1}^{1} \frac{dx}{2\pi} \tilde{F}(x) x^{n-1}$$

In the last expression, we changed the variable as $x = \sin\sigma$ and used the notation $F(\sigma) = \tilde{F}(x)$. Clearly, the Siegel gauge condition requires the vanishing of the function, $\tilde{F}(x) = 0$, therefore $\left|\Psi_0\right\rangle = 0$.

In conclusion, the universal functions cannot be the Siegel gauge.

# References


[1] A. Sen, "Universality of the Tachyon Potential", J. High Energy Phys. **9912** (1999) 027 [arXiv:hep-th/9911116].

[2] A. Sen, "Descent Relations Among Bosonic D-branes", Int. J. Mod. Phys. **A14** (1999) 4061 [arXiv:hep-th/9902105].

[3] A. Sen and B. Zwiebach, "Tachyon Condensation in String Field Theory", J. High Energy Phys. **0003** (2000) 002 [arXiv:hep-th/9912249].

[4] A. Sen, "Tachyon Dynamics in Open String Theory", arXiv:hep-th/0410103, and references therein.

[5] T. Takahashi and S. Tanimoto, "Marginal and Scalar Solutions in Open Cubic String Field Theory", J. High Energy Phys. **0203** (2002) 033 [arXiv:hep-th/0202133].

[6] I. Kishimoto and T. Takahashi, "Open String Field Theory around Universal Solutions", Prog. Theor. Phys. **108** (2002) 591 [arXiv:hep-th/0205275].

[7] Y. Igarashi, K. Itoh, F. Katsumata, T. Takahashi and S. Zeze "Classical Solutions and Order of Zeros in Open String Field Theory", arXiv:hep-th/0502042.

[8] T. Takahashi and S. Zeze, "Gauge Fixing and Scattering Amplitudes in String Field Theory Expanded around Universal Solutions", Prog. Theor. Phys. **110** (2003) 159

[9] T. Takahashi, "Tachyon Condensation and Universal Solutions in String Field Theory", Nucl. Phys. **B670** (2003) 161 [arXiv:hep-th/0302182].

[10] E. Witten, "Non-Commutative Geometry and String Field Theory", Nucl. Phys. **B 268** (1986) 253.

[11] E. Witten, "Interacting Field Theory on Open Superstrings", Nucl. Phys. **B 276** (1986) 291.

[12] A. Leclair, M. Peskin and C. Preitschopf, "String Field Theory on the Conformal Plane (I). Kinematical Principles", Nucl. Phys. **B317** (1989) 411.

[13] A. Leclair, M. Peskin and C. Preitschopf, "String Field Theory on the Conformal Plane (II), Generalized Gluing", Nucl. Phys. **B317** (1989) 464.

[14] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, 1992).

[15] N. Drukker, "On Different for the Vacuum of Bosonic String Field Theory", J. High Energy Phys. **0308** (2003) 017, [arXiv:hep-th/0301079].

[16] S. Zeze, "Worldsheet Geometry of Classical Solutions in String Field Theory," Prog. Theor. Phys. **112** (2004) 863 [arXiv:hep-th/0405097].

[17] N. Drukker, "Closed String Amplitudes from Gauge Fixed String Field Theory", Phys. Rev. **D67** (2003) 126004 [arXiv:hep-th/0207266].

[18] N. Drukker and Y. Okawa, "Vacuum String Field Theory without Matter-ghost Factorization", arXiv:hep-th/0503068.

[19] N. Moeller and W. Taylor, “Level Truncation and the Tachyon in Open Bosonic String Field Theory”, Nucl. Phys. **B538** (2000) 105 [arXiv:hep-th/0002237].

[20] D. Gaiotto and L. Rastelli, “Experimental String Field Theory”, J. High Energy Phys. **0308** (2003) 048 [arXiv:hep-th/0211012].

[21] V. A. Kostelecky and S. Samuel, “On a Nonperturbative Vacuum for the Open Bosonic String”, Nucl. Phys. **B336** (1990) 236.

[22] L. Rastelli and B. Zwiebach, “Tachyon Potentials, Star Products and Universality”, J. High Energy Phys. **09** (2001) 038 [arXiv:hep-th/0006240].

[23] D. Gross and A. Jevicki, “Operator Formalism of Interacting String Field Theory (I)”, Nucl. Phys. **B283** (1987) 1.

[24] D. Gross and A. Jevicki, “Operator Formalism of Interacting String Field Theory (II)”, Nucl. Phys. **B287** (1987) 225.

[25] K. Itoh, K. Ogawa and K. Suehiro, “BRS Invariance of the Witten Type Vertex”, Nucl. Phys. **B289** (1987) 127.

[26] T. Kugo and B. Zwiebach, “Target space duality as a symmetry of string field theory,” Prog. Theor. Phys. **87** (1992) 801 [arXiv:hep-th/9201040].